JOURNAL OF APPROXIMATION THEORY 77, 74–88 (1994)

Linear Widths of Function Spaces Equipped with the Gaussian Measure

VITALY MAIOROV

c/o Allan Pinkus, Technion, Department of Mathematics, Haifa 32000, Israel

Communicated by Allan Pinkus

Received November 20, 1991; accepted in revised form November 20, 1992

We calculate asymptotics for the linear (n, δ)-widths of the Sobolev space W_2^r equipped with the Gaussian measure μ in the L_q. That is, we consider the quantity

$$\lambda_{n,\delta}(W_2^r, L_q, \mu) = \inf_{G \subset W_2^r,\, \mu(G) \leqslant \delta} \lambda_n(W_2^r \backslash G, L_q),$$

where $\lambda_n(K, X)$ is the linear n-width of the set K in the space X.

1. INTRODUCTION

Let X be a normed linear space and W a subset of X. Let Λ be a linear operator from X to X. Let ΛW denote the image of W under Λ. The quantity

$$\lambda(W, \Lambda, X) = \sup_{x \in W} \|x - \Lambda x\|_X$$

is called the linear distance of the image ΛW from the set W.

For each $n = 0, 1, ...,$ we consider the linear n-width of the set W in X. It is defined by

$$\lambda_n(W, X) = \inf_{\mathscr{L}_n} \inf_{\Lambda_n} \lambda(W, \Lambda_n, X),$$

where $\mathscr{L}_n$ runs over all the linear subspaces in X with dimension at most n and Λ_n runs over all linear operators from X to $\mathscr{L}_n$.

We assume that the set W is equipped with a Borel field $\mathscr{B}$, which consists of the open subsets. Let μ be a probability measure defined on $\mathscr{B}$. That is, μ is a σ-additive nonnegative function on $\mathscr{B}$ and $\mu(W) = 1$.

0021-9045/94 $6.00

Let $\delta \in [0, 1]$ be any given number. We define the linear (n, δ)-width of the set W in the space X for the measure μ as follows. Set

$$\lambda_{n,\delta}(W, X, \mu) = \inf_{G_\delta} \lambda_n(W \backslash G_\delta, X), \tag{1}$$

where G_δ runs over all the subsets $G_\delta \in \mathscr{B}$ with measure $\mu(G_\delta) \leqslant \delta$. The quantity $\lambda_{n,\delta}$ may be understood as the μ-distribution of the best linear approximation on all subsets of W.

Detailed information about the usual linear widths may be found in [17, 13]. Papers connected with calculating the asymptotics of linear n-widths include [3, 6, 8, 2].

Quantities similar to (1) were considered in [19]. In the books of Traub and Wozniakowski [18] and Traub *et al.* [19], a different problem connected with the best approximation of the function classes, equipped with measure in a Hilbert space, was investigated. Calculation of n-widths of the smooth function classes equipped with some given measure are included in [20, 9, 1, 11].

Consider the Hilbert space L_2 of all functions $x(t)$, $t \in [0, 2\pi]$, with the Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} C_k \exp(ikt)$$

and inner product

$$\langle x, y \rangle = \frac{1}{2\pi} \int_0^{2\pi} x(t)\, \bar{y}(t)\, dt \qquad (x, y \in L_2).$$

In the space L_2 we define the Veil r-fractional derivative $(r \in \mathbb{R})$

$$x^{(r)}(t) = \sum_{k=-\infty}^{\infty} (ik)^r c_k \exp(ikt) \qquad \left((ik)^r = |k|^r \exp\left(\frac{\pi i}{2} \operatorname{sign} r\right)\right).$$

In this work we consider the Sobolev space W_2^r $(r > 0)$, which consists of all functions $x \in L_2$, with mean value $c_0 = 0$, and semi-norm $\|x\|_{W_2^r} = \langle x^{(r)}, x^{(r)} \rangle$. The space W_2^r is a Hilbert space with the inner product defined by $\langle x, y \rangle_1 = \langle x^{(r)}, y^{(r)} \rangle$.

We equip W_2^r with a Gaussian measure μ whose mean is zero and whose correlation operator C_μ has eigenfunctions $e_k = \exp(ik(\cdot))$ and eigenvalues $\lambda_k = a\,|k|^{-s}$ $(a > 0, s > 1)$. That is, $C_\mu e_k = \lambda_k e_k$, $k \in \mathbb{Z} \backslash \{0\}$.

In particular on the cylindrical subsets in th space W_2^r given by

$$G = \{x \in W_2^r : (\langle x, e_n^{(-r)} \rangle_1, ..., \langle x, e_m^{(-r)} \rangle_1) \in \mathscr{D}\},$$

where $\mathscr{D}$ is any Borel subset in $\mathbb{R}^{m-n+1}$ $(m>n)$, $e_k^{(-r)}=(ik)^{-r}\exp(ik(\cdot))$, $k=\pm 1, \pm 2, \ldots$, is an orthonormal system in W_2^r, and the measure $\mu(G)$ is equal to

$$\mu(G)=\prod_{k=-n}^{m}(2\pi\lambda_k)^{-1/2}\int_{\mathscr{D}}\exp\left(-\tfrac{1}{2}\sum_{k=n}^{m}\lambda_k^{-1}u_k^2\right)du_n\cdots du_m. \tag{2}$$

Detailed information about Gaussian measures may be found in the books of Kuo [5] and Traub *et al.* [19].

Consider the Banach space L_q, $1\leqslant q\leqslant\infty$, which consists of all function x on $[0, 2\pi]$ with norm

$$\|x\|_{L_q}=\left(\int_0^{2\pi}|x(t)|^q\,dt\right)^{1/q}.$$

It is known that if $r>1/2-1/q$, then the space W_2^r is compactly embeddable in the space L_q (see, e.g., [12]).

Let c, c_i, c_i', $i=0, 1, \ldots$ be positive constants depending solely upon the parameter r, q, a, and s. For two positive functions $a(y)$ and $b(y)$, $y\in\mathscr{D}$, we write $a(y)\asymp b(y)$ or $a(y)\ll b(y)$ if there exist constants c_1, c_2, or c such that $c_1\leqslant a(y)/b(y)\leqslant c_2$ or respectively $a(y)\leqslant cb(y)$ for all $y\in\mathscr{D}$.

The aim of this paper is to calculate the asymptotics of the linear (n,δ)-widths $\lambda_{n,\delta}(W_2^r, L_q, \mu)$. Note that the two-sided estimation for $\lambda_{n,\delta}(W_2^r, L_2, \mu)$ may be obtained from the work of Traub *et al.* [19].

THEOREM 1. *Let* $2\leqslant q<\infty$, $r>1/2-1/q$, $s>1$, $a>0$. *The linear* (n,δ)-*widths of* W_2^r *with measure* μ *in the space* L_q *satisfy the asymptotics*

$$\lambda_{n,\delta}(W_2^r, L_q, \mu)\asymp\frac{1+n^{-1/q}\sqrt{\ln\delta^{-1}}}{n^{r+(s-1)/2}},$$

for any $\delta\in(0, \tfrac{1}{2}]$.

We denote the unit ball in W_2^r by $BW_2^r=\{x\in W_2^r\colon \|x\|_{W_2^r}\leqslant 1\}$. The following inequality (see [19, p. 469]) holds for the measure of the unit ball

$$\mu(BW_2^r)>1-5\exp\left(-\frac{1}{2\,\mathrm{trace}\,C_\mu}\right), \tag{3}$$

where $\mathrm{trace}\,C_\mu=\sum_{k=-\infty}^{\infty}\lambda_k=2a\sum_{k=1}^{\infty}k^{-s}$. Therefore for all $a\in I_s\equiv(0, (4\ln 5\zeta(s)^{-1}]$, $\zeta(s)=\sum_{k=1}^{\infty}k^{-s}$ we have from (3) the inequality $\mu(BW_2^r)\geqslant c>0$. We always assume $a\in I_s$.

We define the conditional measure by

$$\mu'(G)=\frac{\mu(G\cap BW_2^r)}{\mu(BW_2^r)}\qquad(G\in\mathscr{B}).$$

We may view μ' as a probability measure defined on the sets Q of the field $\mathscr{B} \cap BW_2^r$ and $\mu'(Q) = \mu(Q)/\mu(BW_2)$.

THEOREM 2. *Let $2 \leqslant q < \infty$, $r > 1/2 - 1/q$, $s > 1$, $a \in I_s$. The following asymptotic equivalence holds for the ball BW_2^r with measure μ' in the space L_q,*

$$\lambda_{n,\delta}(BW_2^r, L_q, \mu') \asymp \min\left\{\frac{1}{n^{r-1/2+1/q}}, \frac{n^{1/q} + \sqrt{\ln \delta^{-1}}}{(n^{r+(s-1)/2+1/q})}\right\}$$

for any $\delta \in (0, \frac{1}{2}]$.

Note that for linear n-widths of the ball BW_2^r in the space L_q we have the equality (see [3])

$$\lambda_n(BW_2^r, L_q) \asymp \frac{1}{n^{r-1/2+1/q}} \qquad (2 \leqslant q \leqslant \infty). \tag{4}$$

Comparing Theorem 2 and the asymptotics of (4) shows, in particular, that if we throw out from the class BW_2^r some set G with measure $\mu'(G) \leqslant \exp(-n^{2/q})$, then we obtain on the remaining set $BW_2^r \backslash G$ the approximation order $n^{-r-(s-1)/2}$. If $s = 1 + 2\varepsilon$, where ε is an arbitrary positive small number, then the approximation order is $n^{-r-\varepsilon}$, which is essentially smaller than (4).

Using Theorem 2 we obtain the asymptotic equivalence for the best approximation on the class BW_2^r in the space L_q by trigonometric polynomials of degree n.

Let $\mathscr{T}_n$ denote the space of trigonometric polynomials of degree n, i.e.,

$$y(t) = \sum_{k=-n}^{n} c_k \exp(ikt).$$

Let the δ-distance from the class BW_2^r to $\mathscr{T}_n$ in the space L_q for the measure μ' be defined by

$$E_{n,\delta}(BW_2^r, L_q, \mu') = \inf_{\{G:\, \mu'(G) \leqslant \delta\}} \sup_{x \in BW_2^r \backslash G} \inf_{y \in \mathscr{T}_n} \|x - y\|_{L_q}.$$

THEOREM 3. *If the conditions of Theorem 2 hold, then*

$$E_{n,\delta}(BW_2^r, L_q, \mu) \asymp \min\left\{\frac{1}{n^{r-1/2+1/q}}, \frac{n^{1/q} + \sqrt{\ln \delta^{-1}}}{n^{r+(s-1)/2+1/q}}\right\}.$$

For $\delta=0$, this result is known (see [12])

$$E_n(BW_2^r, L_q) \asymp \frac{1}{n^{r-1/2+1/q}}. \tag{5}$$

Comparing the asymptotics (5) and Theorem 3 shows, in particular, that on the set $BW_2^r \backslash G$, where G is some set of BW_2^r measure $\mu'(G) \leqslant \exp(n^{-2/q})$, we can obtain an approximation order essentially smaller than (5).

The proofs of Theorems 1–3 use discretisation techniques (see [7]). This method is based on the reduction of the caculation of the widths of a given class, to the calculation of widths of finite-dimensional set equipped with the Gaussian measure.

2. The Estimation of Linear (n, δ)-Widths of Finite-Dimensional Sets

In this section we calculate the linear (n, δ)-widths in the space $\mathbb{R}^m$ equipped with the Gaussian measure in the l_q^m-norm.

Let l_p^m denote the m-dimensional normed space consisting of vectors $x=(x_1, ..., x_m) \in \mathbb{R}^m$ with norm

$$\|x\|_p = \begin{cases} \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p}, & 1 \leqslant p < \infty \\ \max_{1 \leqslant i \leqslant m} |x_i|, & p = \infty. \end{cases}$$

Let $B_p^m(\rho) = \{x \in l_p^m : \|x\|_p \leqslant \rho\}$ be the ball of radius ρ in l_p^m. Set $B_p^m = B_p^m(1)$.

In the space $\mathbb{R}^m$ we consider the Gaussian measure $\gamma = \gamma_m$, which is defined as

$$\gamma(G) = (2\pi)^{-m/2} \int_G \exp\left(-\tfrac{1}{2} \sum_{i=1}^{m} x_i^2 \right) dx_1 \cdots dx_m,$$

where G is any Borel set in $\mathbb{R}^m$. Obviously, $\gamma(\mathbb{R}^m)=1$. We use the following measure estimates for balls (see, e.g., [19]). Namely,

$$\gamma(B_2^m(c\sqrt{m})) \leqslant \tfrac{1}{2}, \qquad \gamma(B_2^m(\rho)) \geqslant 1 - 5\exp\left(-\frac{\rho^2}{2m} \right), \tag{6}$$

where c is an absolute constant, and ρ any positive number.

Consider the linear (n, δ)-widths, with measure, in the l_q^m-norm, namely

$$\lambda_{n,\delta}(\mathbb{R}^m, l_q^m, \gamma) = \inf_{G_\delta} \inf_{A_n} \sup_{x \in \mathbb{R}^m \setminus G_\delta} \|x - A_n x\|_q.$$

where G_δ runs over all Borel sets in $\mathbb{R}^m$ with measure $\gamma(G_\delta) \leqslant \delta$, A_n runs over all linear operators on $\mathbb{R}^m$ with rank at most n, and $\delta \in [0, 1]$.

THEOREM 4. *Let* $2 \leqslant q < \infty$, $m \geqslant 2n > 0$, *and* $\delta \in (0, 1/2]$. *Then*

$$\lambda_{n,\delta}(\mathbb{R}^m, l_q^m, \gamma) \asymp m^{1/q} + \sqrt{\ln \delta^{-1}}.$$

We first prove two auxiliary lemmas. We use some known estimations. Namely, if $u > 1/\sqrt{2}$, then (see [15])

$$\int_u^\infty e^{-t^2}\, dt \leqslant \frac{1}{2u} e^{-u^2}, \qquad \frac{2}{\sqrt{\pi}} \int_0^u e^{-t^2}\, dt \leqslant 1 - \frac{1}{2\sqrt{\pi}} e^{-u^2}. \tag{7}$$

LEMMA 1. *Let* $2 \leqslant q < \infty$ *and* $\delta \in [0, \frac{1}{2}]$. *For some constant* c_0 *depending only on* q, *we have*

$$\gamma(x \in \mathbb{R}^m \colon \|x\|_q \geqslant c_0(m^{1/q} + \sqrt{\ln \delta^{-1}})\} \leqslant \delta. \tag{8}$$

Proof. Let $f(x) = \|x\|_q$. Then $|f(x) - f(y)| \leqslant \|x - y\|_2$; hence f is a Lipschitz map on $\mathbb{R}^m$ with Lipschitz constant $\sigma = 1$. By the Maurey–Pisier inequality, see [14], we have for all $t > 0$,

$$\gamma(f - Ef \geqslant t) \leqslant \exp(-t^2/2\sigma) = \exp(-t^2/2). \tag{9}$$

But by Kahane's inequality, see [4], $E_f \asymp (Ef^q)^{1/q} \asymp q^{1/2} m^{1/q}$. Therefore for some absolute constant $c > 0$.

$$\gamma(f \geqslant t + cq^{1/2} m^{1/q}) \leqslant \exp(-t^2/2), \tag{10}$$

and taking $t = (\ln \delta^{-1})^{1/2}$ completes the proof.

LEMMA 2. *Let* $\delta \in [0, e^{-1}]$. *For any vector* $z \in \mathbb{R}^m$, *we have*

$$\gamma(x \colon |(x, z)| \geqslant c_0' \|z\|_2 \sqrt{\ln \delta^{-1}}) \geqslant \delta,$$

where $(\cdot, \cdot)$ *is the standard inner product, and* c_0' *an absolute constant.*

Proof. Since γ is an invariant measure with respect to orthogonal transformation in the space $\mathbb{R}^m$, it suffices to prove the lemma for the vector $z^* = (\|z\|_2, 0, \ldots, 0)$. Using inequality (7) we have

$$\gamma(x \colon |(x, z^*)| \geqslant \|z\|_2 \sqrt{\ln \delta^{-1}}) \geqslant \frac{1}{2} \sqrt{\frac{\delta}{\pi}}.$$

The lemma now follows.

Proof of Theorem 4. We first prove the upper bound for the linear (n, δ)-widths. Consider the set in $\mathbb{R}^m$:

$$Q_\delta = \{x \in \mathbb{R}^m \colon \|x\|_q \geqslant c_0(m^{1/q} + \sqrt{\ln \delta^{-1}})\}.$$

From Lemma 1 we have the estimation $\gamma(Q_\delta) \leqslant \delta$. Therefore

$$\lambda_{n,\delta} \equiv \lambda_{n,\delta}(\mathbb{R}^m, l_q^m, \gamma) \leqslant \sup_{x \in \mathbb{R}^m \setminus Q_\delta} \|x\|_q \leqslant c_0(m^{1/q} + \sqrt{\ln \delta^{-1}}).$$

To prove the lower bound, let ε be any positive number. We define a linear operator T with rank at most n and a set $G \subset \mathbb{R}^m$ with measure $\gamma(G) \leqslant \delta$, for which

$$\lambda_{n,\delta} \geqslant \sup_{x \in \mathbb{R}^m \setminus G} \|x - Tx\|_q - \varepsilon. \tag{11}$$

We can describe the operator T by

$$Tx = \sum_{k=1}^{n} (x, u_k) v_k \qquad (x \in \mathbb{R}^m),$$

where u_k, v_k are vectors in $\mathbb{R}^m$. We have for $1/q + 1/q' = 1$

$$\begin{aligned} \|x - Tx\|_q &= \max_{y \in B_{q'}^m} (x - Tx, y) = \max_y \left(y - \sum_{k=1}^{n} (y, v_k) u_k, x \right) \\ &\geqslant \max_{1 \leqslant i \leqslant m} \left| \left(e_i - \sum_{k=1}^{n} (e_i, v_k) u_k, x \right) \right|, \end{aligned} \tag{12}$$

where e_i is the ith unit vector. Let $z_i = e_i - \sum_{k=1}^{n} (e_i, v_k) u_k$. Consider the set

$$H = \bigcup_{i=1}^{m} H_i, \qquad H_i = \{x \in \mathbb{R}^m \colon |(x, z_i)| \geqslant c_0' \sqrt{\tfrac{1}{2} \ln \delta^{-1}}\},$$

We know (see [16]) that for any vectors u_k, v_k, $k = 1, ..., n$, there exists an index i_0 such that $\|z_{i_0}\|_2 \geqslant 1/\sqrt{2}$. Therefore from Lemma 2

$$\gamma(H) \geqslant \gamma(H_{i_0}) \geqslant \gamma(x \colon (x, z_{i_0})| \geqslant c_0' \|z_{i_0}\|_2 \sqrt{\ln \delta^{-1}}) > \delta. \tag{13}$$

From inequalities (12) and (13) and since $\gamma(G) \leqslant \delta$, we have

$$\sup_{x \in \mathbb{R}^m \setminus G} \|x - Tx\|_q \geqslant c_0' \sqrt{\tfrac{1}{2} \ln \delta^{-1}}.$$

From here and inequality (11), letting $\varepsilon \to 0$, we obtain

$$\lambda_{n,\delta} \geqslant c_0' \sqrt{\tfrac{1}{2} \ln \delta^{-1}}. \tag{14}$$

We obtain one more lower estimate for $\lambda_{n,\delta}$. Using Hölder's inequality we have

$$\lambda_{n,\delta} \geqslant m^{1/q - 1/2} \lambda_{n,\delta}(\mathbb{R}^m, l_2^m, \gamma) = m^{1/q - 1/2} \inf_{G:\, \gamma(G) \leqslant \delta} \sup_{x \in \mathbb{R}^m \setminus G} p(x),$$

where $p(x) = (\sum_{i=1}^{m-n} x_i^2)^{1/2}$. From inequality (6) it follows that

$$\gamma(p(x) \geqslant c\sqrt{m}) = \gamma_{m-n}(\mathbb{R}^{m-n} \setminus B_2^{m-n}(c\sqrt{m-n})) \geqslant \tfrac{1}{2}.$$

Therefore, for any $\delta < \frac{1}{2}$ using $m \geqslant 2n$, we obtain

$$\lambda_{n,\delta} \geqslant m^{1/q - 1/2} c\sqrt{m-n} \geqslant \frac{c}{\sqrt{2}} m^{1/q}. \tag{15}$$

From inequalities (14) and (15) we obtain the lower estimate for $\lambda_{n,\delta}$, and Theorem 4 is proved.

Consider in the space $\mathbb{R}^m$ the Gaussian measures with parameter α given by

$$\bar{\gamma} = \bar{\gamma}_m = \left(\frac{\alpha}{2\pi}\right)^{m/2} \int_G \exp\left(-\frac{\alpha}{2} \sum_{i=1}^m x_i^2\right) dx_1 \cdot \cdots \cdot dx_m.$$

Set $B = B_2^m(\sqrt{m})$. From inequality (6) for $\alpha > \alpha_0 \equiv 1 + 2\ln 10$ we have

$$\bar{\gamma}(B) \geqslant \tfrac{1}{2}.$$

Define on $\mathbb{R}^m$ the conditional measure concentrated on the ball B, i.e.,

$$\bar{\gamma}'(G) = \frac{\bar{\gamma}(G \cap B)}{\bar{\gamma}(B)} \qquad (G \subset \mathbb{R}^m).$$

Obviously $\bar{\gamma}'(B) = 1$. Therefore $\bar{\gamma}'$ is a probability measure defined on the Borel subsets of the ball B and

$$\bar{\gamma}'(G) = \frac{\bar{\gamma}(G)}{\bar{\gamma}(B)} \qquad (G \subset B)$$

LEMMA 2A. *If $\delta \in [\exp(-m/2), 1/2]$ and $z \in \mathbb{R}^m$, then*

$$\bar{\gamma}'(x \in B\colon |(x, z)| \geqslant c' \|z\|_2 \sqrt{\tfrac{1}{2} \ln \delta^{-1}}) \geqslant \delta, \tag{16}$$

where c' is an absolute constant.

Indeed from the fact that $\bar{\gamma}'$ is an invariant measure with respect to orthogonal transposition, it follows that

$$g \equiv \bar{\gamma}(x \in \mathbb{R}^m: |(x, z)| \geqslant \|z\|_2 \sqrt{\ln \delta^{-1}}, \|x\|_2 \leqslant \sqrt{m})$$
$$= 2\left(\frac{\alpha}{2\pi}\right)^{m/2} \int_{\sqrt{\ln \delta^{-1}}}^{\sqrt{m}} e^{-(\alpha/2)x_1^2}\, dx_1$$
$$\times \int_{B_2^{m-1}(\sqrt{m - x_1^2})} \exp\left(-\frac{\alpha}{2}(x_2^2 + \cdots + x_m^2)\right) dx_2 \cdots dx_m.$$

Further, from the definitions of δ and α we have

$$g \geqslant 2\left(\frac{\alpha}{2\pi}\right)^{m/2} \int_{\sqrt{\ln \delta^{-1}}}^{\sqrt{3/2 \ln \delta^{-1}}} e^{-(\alpha/2)x_1^2}\, dx_1$$
$$\times \int_{B_2^{m-1}(\sqrt{m - 3/2 \ln \delta^{-1}})} \exp\left(-\frac{\alpha}{2}(x_2^2 + \cdots + x_m^2)\right) dx_2 \cdots dx_m$$
$$\geqslant \sqrt{\frac{2\alpha}{\pi}}\, \delta^{3\alpha/4} \cdot \bar{\gamma}_{m-1}(B_2^{m-1}(c_1 \sqrt{m})), \tag{17}$$

where $c_1 = \frac{1}{2}$. From inequality (6) it follows that $\bar{\gamma}_{m-1}(B_2^{m-1}(c_1 \sqrt{m})) \geqslant c_0'$, where c_0' is an absolute constant. Hence using inequality (17) and the definition of α we have

$$\bar{\gamma}'(x \in B: |(x, z)| \geqslant \|z\|_2 \sqrt{\ln \delta^{-1}}) \geqslant g \geqslant \sqrt{\frac{2\alpha}{\pi}}\, c_0'\, \delta^{3\alpha/4} \geqslant c_2 \delta^{3\alpha/4}.$$

Inequality (16) now follows.

COROLLARY 1. *Let $2 \leqslant q < \infty$ and $\alpha > \alpha_0$. For the linear (n, δ)-widths of the ball B, with measure $\bar{\gamma}'$, in the space l_q^m we have*

$$\lambda_{n,\delta}(B, l_q^m, \bar{\gamma}') \asymp \min\{\sqrt{m}, \alpha^{-1/2}(m^{1/q} + \sqrt{\ln \delta^{-1}})\}, \tag{18}$$

where $m \geqslant 2n$ and $\delta \in [0, \frac{1}{2}]$.

The upper estimate in (18) follows directly from Theorem 4 and the obvious inequality $\lambda_{n,\delta}(B, l_q^m, \bar{\gamma}') \leqslant \sqrt{m}$. The lower estimate in (18) for $\delta \geqslant \delta_0$, $\delta_0 = \exp(-\alpha m)$, repeats the proof of the lower estimate in Theorem 4. However, we must use Lemma 2A rather than Lemma 2. Then we have

$$\lambda_{n,\delta}(B, l_q^m, \bar{\gamma}') \geqslant c\alpha^{-1/2}(m^{1/q} + \sqrt{\ln \delta^{-1}}). \tag{19}$$

If $\delta < \delta_0$, then from (19) we have

$$\lambda_{n,\delta}(B, l_q^m, \bar{\gamma}') \geq \lambda_{n,\delta_0}(B, l_q^m, \bar{\gamma}') \geq c\sqrt{m}. \tag{20}$$

From the inequalities (19) and (20) we obtain (18).

3. Proof of Theorem 1

First we give a few auxiliary statements. Consider two sequences of integers $m_0 = 0$, $m_N = 3^{N-1}$ and $l_0 = 0$, $l_N = \sum_{s=1}^{N} m_s$, where $N = 1, 2, \ldots$. We decompose the integers $\mathbb{Z}$ on blocks $\{\Delta_N\}_{N=-\infty}^{\infty}$, where $\Delta_0 = \{0\}$, $\Delta_N = \{l_N, \ldots, l_{N+1} - 1\}$ for $N = 1, 2, \ldots$, and $\Delta_N = -\Delta_{-N}$ for $N = -1, -2, \ldots$. For negative N, set $m_N = m_{-N}$, $l_N = l_{-N}$. The cardinality of the block Δ_N equals m_N.

For any number N we denote T_N the space consisting of the trigonometric series $y(\cdot) = \sum_{k \in \Delta_N} c_k \exp(ik(\cdot))$. We define on the space T_N the norm

$$\|y\|_{q,N} = \left(\sum_{s=0}^{m_N - 1} \left| y\left(\frac{2\pi s}{m_N}\right)\right|^q\right)^{1/q}.$$

From the Hardy–Littlewood inequality (see [21, Vol. 2, p. 4]) for any $1 < q < \infty$ we have

$$\|y\|_{L_q} \asymp m_N^{-1/q} \|y\|_{q,N} \qquad (y \in T_N). \tag{21}$$

In particular for $q = 2$ we have the equality $\|y\|_{L_2} = m_N^{-1/2} \|y\|_{2,N}$. From the Marcinkiewicz theorem (see [21]) we have

$$\|y^{(\alpha)}\|_{L_q} \asymp m^{\alpha} \|y\|_{L_q} \qquad (y \in T_N, \alpha \in \mathbb{R}). \tag{22}$$

For every N we consider, in the space W_2^r, the projection operator

$$P_N\colon \sum_{k=-\infty}^{\infty} c_k e_k \to \sum_{k \in \Delta_N} c_k e_k. \tag{23}$$

Lemma 3. *Set*

$$\alpha_{N,\delta} = \frac{c_0'(m_N^{1/q} + \sqrt{\ln \delta^{-1}})}{m_N^{r + 1/q + (s-1)/2}},$$

where c_0' is some constant depending just on r and q. Then for any $\delta \in [0, \frac{1}{2}]$ we have

$$\mu(x \in W_2^r\colon \|P_N x\|_{L_q} \geq \alpha_{N,\delta}) \leq \delta.$$

Proof. Since

$$P_N x = \sum_{k \in \Delta_N} \langle x, e_k \rangle e_k = \sum_{k \in \Delta_N} \langle x^{(-r)}, e_k^{(-r)} \rangle_1 \cdot e_k,$$

from (22) and from the definition of Gaussian measure (2) we have

$$\mu \equiv \mu(x\colon \|P_N x\|_{L_q} \geqslant \alpha_{N,\delta}) \leqslant \mu\left(x\colon \left\| \sum_{k \in \Delta_N} \langle x, e_k^{(-r)} \rangle_1 e_k \right\|_{L_q} \geqslant c_1 m_N^r \alpha\right)$$
$$= \left(\prod_{k=l_N}^{l_{N+1}-1} 2\pi\lambda_k \right)^{-1/2} \int_{\mathscr{D}} \exp\left(-\tfrac{1}{2} \sum_{k=l_N}^{l_{N+1}-1} \lambda_k^{-1} y_k \right) dy_{l_N} \cdots dy_{l_{N+1}-1},$$

where

$$\mathscr{D} = \left\{ (y_{l_N}, ..., y_{l_{N+1}-1}) \in \mathbb{R}^{m_N}\colon \left\| \sum_{k=l_N}^{l_{N+1}-1} y_k e_k \right\|_{L_q} \geqslant c_1 m_N^r \alpha_{N,\delta} \right\}.$$

Recall that $\lambda_k = a\,|k|^{-s}$. By the substitution $t_k/\sqrt{\lambda_k} \to t_k$ and equality (22) we have

$$\mu \leqslant (2\pi)^{-m_N/2} \int_{\mathscr{D}_1} \exp\left(-\tfrac{1}{2} \sum_{k=1}^{m_N} y_k \right) dy_1 \cdots dy_{m_N} = \gamma_{m_N}(\mathscr{D}_1), \tag{24}$$

where $\mathscr{D}_1 = \{ y \in \mathbb{R}^{m_N}\colon \|\sum_{k=1}^{m_N} y_k e_k\|_{L_q} \geqslant c_2 m_N^{r+s/2} \}$.

From (21) it follows that the set $\mathscr{D}_1$ is contained in the set

$$\mathscr{D}_2 = \left\{ y \in \mathbb{R}^{m_N}\colon \left(\sum_{u=1}^{m_N} \left| \sum_{k=1}^{m_N} y_k e_k \left(\frac{2\pi u}{m_N} \right) \right|^q \right)^{1/q} \geqslant c_3 m_N^{r+s/2+1/q} \alpha_{N,\delta} \right\}.$$

Since the matrix $(e_k(2\pi u/m_N)/\sqrt{m_N})_{k,u=1,...,m_N}$ is orthogonal, and finite-dimensional Gaussian measure is invariant with respect to orthogonal transposition in the space $\mathbb{R}^{m_N}$, we have

$$\gamma_{m_N}(\mathscr{D}_1) \leqslant \gamma_{m_N}(\mathscr{D}_2) = \gamma_{m_N}(\mathscr{D}_3), \tag{25}$$

where $\mathscr{D}_3 = \{ y \in \mathbb{R}^{m_N}\colon (\sum_{k=1}^{m_N} |y_k|^q)^{1/q} \geqslant c_3 m_N^{r+(s-1)/2+1/q} \alpha_{N,\delta} \}$.

Let $c_0' = c_0/c_3$, where c_0 is the constant from Lemma 1. Using the definition $\alpha_{N,\delta}$ from Lemma 1, we obtain

$$\gamma_{m_N}(\mathscr{D}_3) = \gamma_{m_N}(y \in \mathbb{R}^{m_N}\colon \|y\|_q \geqslant c_0(m_N^{1/q} + \sqrt{\ln \delta^{-1}})) \leqslant \delta. \tag{26}$$

From inequalities (24)–(26) we obtain Lemma 3.

Proof of Theorem 1. We first prove the upper bound for linear (n, δ)-widths $\lambda_{n,\delta}(W_2^r, L_q, \mu)$. Denote $\delta_N = \delta \cdot 3^{(N'-|N|)}$, $N = \pm N', \pm(N'+1), \ldots,$ $N' = [\log n/2]$ (where [ ] is the integer part of number and $\log u = \log_3 u$). Consider the sequence of sets

$$G_N = \{x \in W_2^r : \|P_N x\|_{L_q} \geqslant \alpha_{N,\delta_N}\}.$$

Using Lemma 3 we estimate the measure of the set $G = \bigcup_{|N| \geqslant N'} G_N$

$$\mu(G) \leqslant \sum_{|N| \geqslant N'} \mu(G_N) \leqslant \sum_{|N| \geqslant N'} \delta_N \leqslant \delta \sum_{|N| \geqslant N'} 3^{N'-|N|} \ll \delta. \tag{27}$$

Since $m_N = 3^{|N|}$, we have

$$\begin{aligned}\sum_{|N| \geqslant N'} \alpha_{N,\delta_N} &= c_0' \sum_{|N| \geqslant N'} \frac{m_N^{1/q} + \sqrt{\ln \delta_N^{-1}}}{m_N^{r+1/q+(s-1)/2}} \\ &\ll \sum_{|N| \geqslant N'} 3^{-[r+(s-1)/2]|N|}(1 + 3^{-|N|/q} \sqrt{\ln \delta_N^{-1}}) \\ &\ll 3^{-[r+(s-1)/2]N'}(1 + 3^{-N'/q} \sqrt{\ln \delta^{-1}}) \\ &\ll \frac{1 + n^{-1/q}\sqrt{\ln \delta^{-1}}}{n^{r+(s-1)/2}}.\end{aligned} \tag{28}$$

Consider the linear operator $A_n = \sum_{N=-N'}^{N'} P_N$. From the inequalities (27) and (28) we have for linear (n, δ)-widths the estimate

$$\begin{aligned}\lambda_{n,\delta}(W_2^r, L_q, \gamma) &\leqslant \lambda(W_2^r \backslash G, A_n, L_q) \\ &\ll \sup_{x \in W_2^r \backslash G} \left\| \sum_{|N| \geqslant N'} P_N x \right\|_{L_q} \\ &\leqslant \sum_{|N| \geqslant N'} \alpha_{N,\delta_N} \ll \frac{1 + n^{-1/q}\sqrt{\ln \delta^{-1}}}{n^{r+(s-1)/2}}.\end{aligned}$$

We now prove the lower estimate. Let ε be any prositive number. Denote by A_n the linear operator of rank at most n, and by G_δ the set in W_2^r with measure $\mu(G_\delta) \leqslant \delta$ such that

$$\lambda_{n,\delta}(W_2^r, L_q, \mu) \geqslant \lambda(W_2^r \backslash G_\delta, A_n, L_q) - \varepsilon. \tag{29}$$

Consider the projection operator in the space L_q given by $Q_n : \sum_{k=-\infty}^{\infty} c_k e_k \to \sum_{k=n}^{3n+1} c_k e_k$. From the Marcinkewicz theorem we have the inequality $\|Q_n x\|_{L_q} \leqslant c_1 \|x\|_{L_q}$ for all $x \in L_q$. Further, using (22) and the definition of the Gaussian measure (2) we have

$$\begin{aligned}
&\mu(x\in W_2^r\colon \|x-A_nx\|_{L_q}\geqslant c_1^{-1}\alpha_{N,\delta})\\
&\quad\geqslant \mu(\|Q_nx-Q_nA_nx\|_{L_q}\geqslant \alpha_{n,\delta})\\
&\quad=\mu\left(\left\|\sum_{k=n}^{3n+1}\langle x,e_k\rangle(e_k-Q_nA_ne_k)\right\|_{L_q}\geqslant\alpha_{n,\delta}\right)\\
&\quad\geqslant\mu\left(\left\|\sum_{k=n}^{3n+1}\langle x,e_k^{(-r)}\rangle_1(e_k-Q_nA_ne_k)\right\|_{L_q}\geqslant c_2n^r\alpha_{n,\delta}\right)\\
&\quad=\left(\prod_{k=n}^{3n+1}\frac{1}{2\pi\lambda_k}\right)^{1/2}\int_{\mathscr{D}}\exp\left(-\frac{1}{2}\sum_{k=n}^{3n+1}\lambda_k^{-1}y_k^2\right)dy_n\cdots dy_{3n+1},
\end{aligned}\tag{30}$$

where

$$\mathscr{D}=\left\{y=(y_n,\dots,y_{3n+1})\in\mathbb{R}^{2n+1}\colon\left\|\sum_{k=n}^{3n+1}y_k(e_k-Q_nA_ne_k)\right\|_{L_q}\geqslant c_2n^r\alpha_n\right\}.$$

With the help of the substitution $t_k/\sqrt{\lambda_k}\to t_k$ and equality (21), we obtain

$$\begin{aligned}
&\left(\prod_{k=n}^{3n+1}2\pi\lambda_k\right)^{-1/2}\int_{\mathscr{D}}\exp\left(-\tfrac{1}{2}\sum_{n}^{3n+1}\lambda_k^{-1}y_k\right)dy_n\cdots dy_{3n+1}\\
&\quad\geqslant\left(\prod_{k=0}^{2n}2\pi\right)^{-1/2}\int_{\mathscr{D}_1}\exp\left(-\tfrac{1}{2}\sum_{k=0}^{2n}y_k^2\right)dy_0\cdots dy_{2n}=\gamma_{2n+1}(\mathscr{D}_1),
\end{aligned}\tag{31}$$

where

$$\mathscr{D}_1=\left\{y\in\mathbb{R}^{2n+1}\colon\left(\sum_{l=0}^{2n}\left|\sum_{k=0}^{2n}y_k[e_k(\theta_l)-(Q_nA_ne_k)(\theta_l)]\right|^q\right)^{1/q}\geqslant c_3n^{r+s/2+1}\right\}$$

and $\theta_l=2\pi l/(2n+1)$. Consider the two matrices

$$E=\left(\frac{e_k(\theta_l)}{\sqrt{2n+1}}\right)_{k,l=0,\dots,2n}\qquad\text{and}\qquad H=\left(\frac{Q_nA_ne_k(\theta_l)}{\sqrt{2n+2}}\right)_{k,l=0,\dots,2n}.$$

Then we can write

$$\begin{aligned}
\mathscr{D}_1&=\{y\in\mathbb{R}^{2n+1}\colon\|Ey-Hy\|_q\geqslant c_3n^{r+(s-1)/2+1/q}\alpha_{n,\delta}\}\\
&=\{y\in\mathbb{R}^{2n+1}\colon\|Ey-Hy\|_q\geqslant c_3c_0'(n^{1/q}+\sqrt{\ln\delta^{-1}})\}.
\end{aligned}$$

Since E is an orthogonal matrix and the finite-dimensional Gaussian measure γ_{2n+1} is invariant with respect to orthogonal transposition in the space $\mathbb{R}^{2n+1}$, we have

$$\gamma_{2n+1}(\mathscr{D}_1)=\gamma_{2n+1}(\mathscr{D}_2),\tag{32}$$

where

$$\mathscr{D}_2 = \{ y \in \mathbb{R}^{2n+1} : \| y - HE^{-1} y \|_q \geqslant c_3 c_0' (n^{1/q} + \sqrt{\ln \delta^{-1}}) \}.$$

From the definition of the matrix H it follows that the rank of HE^{-1} is at most n. Therefore from Theorem 4, for some constant c_0', we have $\gamma_{2n+1}(\mathscr{D}_2) > \delta$. Further from (30)–(32) it follows that

$$\mu(x \in W_2^r : \| x - \Lambda_n x \|_{L_q} \geqslant c_1^{-1} \alpha_{n,\delta}) \geqslant \gamma_{2n+1}(\mathscr{D}_2) > \delta.$$

Hence we obtain

$$\lambda(W_2^r \backslash G_\delta, \Lambda_n, L_q) \geqslant c_1^{-1} \alpha_{n,\delta}.$$

Letting $\varepsilon \to 0$ in inequality (29) we obtain the lower estimate. Theorem 1 is proved.

The proof of Theorem 2 is analogous to the proof of Theorem 1. Indeed, the upper estimate for the linear (n, δ)-width follows from the inequality

$$\lambda_{n,\delta}(BW_2^r, L_q, \mu') \leqslant \lambda_{n,\delta}(W_2^r, L_q, \mu) \ll \frac{1 + n^{-1/q} \sqrt{\ln \delta^{-1}}}{n^{r + (s-1)/2}}$$

and from the known estimate for linear n-widths (see [3])

$$\lambda_{n,\delta}(BW_2^r, L_q, \mu') \leqslant \lambda_{n,0}(BW_2^r, L_q, \mu') \ll n^{-r + 1/2 - 1/q}.$$

The proof of the lower estimate repeats the proof in Theorem 1. But here, instead of Theorem 4 we use Corollary 1.

The proof of Theorem 3 repeats the proof of Theorem 1. But in the lower estimate, instead of the operator Λ_n, we consider the zero operator.

We remark that an announcement about results on Kolmogorov (n, δ)-widths of the W_2^r spaces with measure μ in the L_q-norm appears in Maiorov [10, 22]. There also appear analogous results for Wiener spaces.

References

1. A. E. Buslaev, About best approximation of random functions and functionals, *in* "Trud. of Th'd Saratov Winter School," Vol. 2, pp. 14–17, Saratov University, 1988.
2. K. Hollig, Approximationszahlen von Sobolev-Einbettungen, dissertation, Bonn, 1979.
3. R. S. Ismagilov, Diameters of sets in normed linear spaces and approximation of functions with trigonometric polynomials, *Uspekhi Mat. Nauk* **29** (1974), 161–178.
4. J. P. Kahane, Some random series of functions, *in* "Heath Math. Monograhps," Heath, Lexington, 1968.
5. H. H. Kuo, Gaussian measures in Banach spaces, *in* "Lecture Notes in Mathematics," Vol. 463, Springer-Verlag, Berlin, 1975.

6. V. E. MAIOROV, About linear widths of Sobolev classes, *Dokl. Akad. Nauk UzSSR* **243** (1978), 1127–1130.
7. V. E. MAIOROV, Discretisation of a problem about n-widths, *Uspekhi Mat. Nauk* **30** (1976), 179–180.
8. V. E. MAIOROV, About linear diameters of Sobolev classes and chains of extremal subspaces, *Mat. Sb.* **113** (1980), 437–463.
9. V. E. MAIOROV, "About Widths of Space, Equipped with a Measure," pp. 91–105, Yaroslav. State University, Yaroslavle, 1990.
10. V. E. MAIOROV, Widths of spaces, equipped with a Gaussian measure, *Dokl. Akad. Nauk UzSSR* **323** (1992), 233–237.
11. P. MATHE, s-number in information-based complexity, *J. Complexity* **6** (1990), 41–66.
12. S. M. NIKOLSKI, "Approximation of the Many Variables Functions and Theorems of Embedding," Nauka, Moscow, 1969.
13. A. PINKUS, "n-Widths in Approximation Theory," Springer-Verlag, Berlin, 1985.
14. G. PISIER, Probabilistic methods in the geometry of Banach spaces, *in* "Lecture Notes in Mathematics," Vol. 1206, pp. 167–241, Springer, New York/Berlin, 1986.
15. G. E. SHILOV AND FAN DYK TIN, "Integral Measure and Derivative on Linear Spaces," Nauka, Moskow, 1967, [In Russian].
16. S. B. STECHKIN, About best approximation of the given function classes with any polynomials, *Uspekhi Mat. Nauk* **9** (1954), 133–134.
17. V. M. TIKHOMIROV, "Some Problems in Approximation Theory," Moscow State University, Moscow, 1976. [In Russian]
18. J. F. TRAUB AND H. WOZNIAKOWSKI, "A General Theory of Optimal Algorithm," Academic Press, New York, 1980.
19. J. F. TRAUB, G. W. WASILKOWSKI, AND H. WOZNIAKOWSKI, "Information-Based Complexity," Academic Press, New York, 1988.
20. S. M. VORONIN AND N. TEMIRGALIEV, About some applications of Banach measure, *Izv. Akad. Nauk Kasakh. SSR Ser. Fiz.-Mat.* **5** (1984), 8–11.
21. A. ZYGMUND, "Trigonometric Series," Vol. 2, Cambridge Univ. Press, London/New York, 1959. [In Russian]
22. V. E. MAIOROV, Kolmogorov (n, δ)-widths of the spaces of the smooth functions, *Math. Sbornik* **184** (1993), 49–70.